B.Sc-part-I(H), paper-I 27-07-2020

Q.1. Define subgroups and a non-empty subset H of a group G is a subgroup of G if and only if (i) $a, b \in H \Rightarrow ab \in H$ (ii) $a \in H \Rightarrow a^{-1} \in H$, where a^{-1} is the inverse of $a \in G$.

Soln: Subgroups:→ A non-empty subset H of a group G is called a subgroup of G if H itself a group with respect to the operation defined in G.

proof:→ Let H be a subgroup of G, then H must be closed with respect to multiplication, i.e. the composition in G.

Therefore, $a \in H ; b \in H \Rightarrow ab \in H$

Conversely:→ Let us suppose H is a subset of a group G such that (I) and (II) the given conditions, hold. In order to show that H is a subgroup, all that is needed is to verify that the identity element $e \in H$ and that the associative law holds for elements of H.

If $a \in H$, then by (I), $a^{-1} \in H$ and so by (I), we see that $e = aa^{-1} \in H$, again since associative law does holds in G, it holds all the more so for H, which is a subset of G. Hence, is a subgroup of G.

Q.2 Let H be a non-empty subset of a group G. Then, H is a subgroup of G.

~~proof~~ $a, b \in H \Rightarrow ab^{-1} \in H$, where b^{-1} is the inverse of b in G.

proof. Necessary conditions: Let us suppose that H is a subgroup of G and $a, b \in H$. Since, H is a group, Rest part of Q.2. 27-07-2020.

each element of H must have its inverse in H. Thus, if $b \in H \Rightarrow b^{-1} \in H$ and then by closure property $ab^{-1} \in H$. This proves the necessary condition.

Sufficient condition:→

Conversely, let H is a subset of G for which $a, b \in H$ implies that $ab^{-1} \in H$. We have to show that H is a subgroup of G, we must verify that H is closed, the identity element $e \in H$, every element of H has an inverse in H and the associative law holds for elements of H.

Let $b = a$, then we see that $a \in H \Rightarrow aa^{-1} \in H \Rightarrow e \in H$.

$\Rightarrow$ identity element of G also belongs to H.

Now, for the elements e and b of H, we have $eb^{-1} \in H$ and so $b^{-1} \in H$, since b is an arbitrary element of H, we see that for any $b \in H$, $b^{-1} \in H$.

Now, $a, b \in H \Rightarrow ab^{-1} \in H$

$\Rightarrow a(b^{-1})^{-1} \in H$ [By hypothesis]

$\Rightarrow ab \in H$

$\Rightarrow$ H is closed.

Finally, since the associative law does hold for H, it also holds for H which is a subset of G.

$\Rightarrow (H, o)$ is a subgroup.

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